Narrowing the Complexity Gap for Colouring (C_s, P_t)-Free Graphs*

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Abstract. For a positive integer k and graph $G = (V, E)$, a *k-colouring* of G is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. The k-Colouring problem is to decide, for a given G, whether a k-colouring of G exists. The k-Precolouring Extension problem is to decide, for a given $G = (V, E)$, whether a colouring of a subset of V can be extended to a k-colouring of G. A k-list assignment of a graph is an allocation of a list — a subset of $\{1, \ldots, k\}$ — to each vertex, and the List k-Colouring problem is to decide, for a given G, whether G has a k-colouring in which each vertex is coloured with a colour from its list. We continued the study of the computational complexity of these three decision problems when restricted to graphs that do not contain a cycle on s vertices or a path on t vertices as induced subgraphs (for fixed positive integers s and t).

1 Introduction

Let $G = (V, E)$ be a graph. A *colouring* of G is a mapping $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. We call $c(u)$ the *colour* of u. A *k-colouring* of G is a colouring with $1 \leq c(u) \leq k$ for all $u \in V$. We study the following decision problem:

k-Colouring
Instance : A graph G.
Question : Is G k-colourable?

It is well-known that k-Colouring is NP-complete even if $k = 3$ [22], and so the problem has been studied for special graph classes; see the surveys of Randerath and Schiermeyer [25] and Tuza [27], and the very recent survey of Golovach, Johnson, Paulusma and Song [9]. In this paper, we consider graph classes defined in terms of forbidden induced subgraphs, and study the computational

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complexity of k-Colouring and some related problems that we introduce now before stating our results.

A *k-precolouring* of $G = (V, E)$ is a mapping $c_W : W \to \{1, 2, \dots k\}$ for some subset $W \subseteq V$. A k-colouring c is an *extension* of c_W if $c(v) = c_W(v)$ for each $v \in W$.

k-Precolouring Extension
Instance: A graph G and a k-precolouring c_W of G.
Question: Can c_W be extended to a k-colouring of G?

A *list assignment* of a graph $G = (V, E)$ is a function L that assigns a list $L(u)$ of *admissible* colours to each $u \in V$. If $L(u) \subseteq \{1, \dots, k\}$ for each $u \in V$, then L is also called a *k-list assignment*. A colouring c *respects* L if $c(u) \in L(u)$ for all $u \in V$. Here is our next decision problem:

List k-Colouring
Instance: A graph G and a k-list assignment L for G.
Question: Is there a colouring of G that respects L?

Note that k-Colouring can be viewed as a special case of k-Precolouring Extension which is, in turn, a special case of List k-Colouring.

A graph is (C_s, P_t)-free if it has no subgraph isomorphic to either C_s, the cycle on s vertices, or P_t, the path on t vertices. Several papers [4, 10, 14] have considered the computational complexity of our three decision problems when restricted to (C_s, P_t)-free graphs. In this paper, we continue this investigation. Our first contribution is to state the following theorem that provides a complete summary of our current knowledge. The cases marked with an asterisk are new results presented in this paper. We use *p-time* to mean polynomial-time throughout the paper.

Theorem 1. *Let k, s, t be three positive integers. The following statements hold for (C_s, P_t)-free graphs.*

(i) List k-Colouring *is* NP*-complete if*

*1.** $k \geq 4$, $s = 3$ *and* $t \geq 8$
*2.** $k \geq 4$, $s \geq 5$ *and* $t \geq 6$.

List k-Colouring *is p-time solvable if*

3. $k \leq 2$, $s \geq 3$ *and* $t \geq 1$
4. $k = 3$, $s = 3$ *and* $t \leq 6$
5. $k = 3$, $s = 4$ *and* $t \geq 1$
6. $k = 3$, $s \geq 5$ *and* $t \leq 6$
7. $k \geq 4$, $s = 3$ *and* $t \leq 6$
8. $k \geq 4$, $s = 4$ *and* $t \geq 1$
9. $k \geq 4$, $s \geq 5$ *and* $t \leq 5$.

(ii) k-Precolouring Extension *is* NP*-complete if*

1. $k = 4$, $s = 3$ *and* $t \geq 10$
2. $k = 4$, $s = 5$ *and* $t \geq 7$

3. $k = 4$, $s = 6$ *and* $t \geq 7$
*4.** $k = 4$, $s = 7$ *and* $t \geq 8$
5. $k = 4$, $s \geq 8$ *and* $t \geq 7$
6. $k \geq 5$, $s = 3$ *and* $t \geq 10$
*7.** $k \geq 5$, $s \geq 5$ *and* $t \geq 6$.

k-Precolouring Extension *is p-time solvable if*

8. $k \leq 2$, $s \geq 3$ *and* $t \geq 1$
9. $k = 3$, $s = 3$ *and* $t \leq 6$
10. $k = 3$, $s = 4$ *and* $t \geq 1$
11. $k = 3$, $s \geq 5$ *and* $t \leq 6$
12. $k \geq 4$, $s = 3$ *and* $t \leq 6$
13. $k \geq 4$, $s = 4$ *and* $t \geq 1$
14. $k \geq 4$, $s \geq 5$ *and* $t \leq 5$.

(iii) k-Colouring *is* NP*-complete if*

*1.** $k = 4$, $s = 3$ *and* $t \geq 22$
2. $k = 4$, $s = 5$ *and* $t \geq 7$
3. $k = 4$, $s = 6$ *and* $t \geq 7$
4. $k = 4$, $s = 7$ *and* $t \geq 9$
5. $k = 4$, $s \geq 8$ *and* $t \geq 7$
*6.** $k \geq 5$, $s = 3$ *and* $t \geq t_k$ *where* t_k *is a constant that only depends on* k
7. $k \geq 5$, $s = 5$ *and* $t \geq 7$
8. $k \geq 5$, $s \geq 6$ *and* $t \geq 6$.

k-Colouring *is p-time solvable if*

9. $k \leq 2$, $s \geq 3$ *and* $t \geq 1$
10. $k = 3$, $s = 3$ *and* $t \leq 7$
11. $k = 3$, $s = 4$ *and* $t \geq 1$
12. $k = 3$, $s \geq 5$ *and* $t \leq 7$
13. $k = 4$, $s = 3$ *and* $t \leq 6$
14. $k = 4$, $s = 4$ *and* $t \geq 1$
15. $k = 4$, $s = 5$ *and* $t \leq 6$
16. $k = 4$, $s \geq 6$ *and* $t \leq 5$
17. $k \geq 5$, $s = 3$ *and* $t \leq k + 2$
18. $k \geq 5$, $s = 4$ *and* $t \geq 1$
19. $k \geq 5$, $s \geq 5$ *and* $t \leq 5$.

The new results on List k-Colouring, k-Precolouring Extension and k-Colouring are in Sections 2, 3 and 4 respectively. In these three sections we often prove stronger statements, for example, on (chordal) bipartite graphs, to strengthen existing results in the literature as much as we can. In Section 5, we prove Theorem 1 by combining a number of previously known results with our new results, and in Section 6 we summarize the open cases and pose a number of related open problems.

We introduce some more terminology that we will need. Let $G = (V, E)$ be a graph. The *chromatic number* of G is the smallest integer k for which G has a k-colouring. Let $\{H_1, \ldots, H_p\}$ be a set of graphs. We say that G is $(H_1, \ldots, H_p)$-*free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we write H_1-free instead of (H_1)-free. The *complement* of G, denoted by $\overline{G}$, has vertex set V and an edge between two distinct vertices if and only if these vertices are not adjacent in G. The disjoint union of two graphs G and H is denoted $G + H$, and the disjoint union of r copies of G is denoted rG. The *girth* of G is the number of vertices of a shortest cycle in G or infinite if G has no cycle. Note that a graph has girth at least g for some $g \geq 4$ if and only if it is $(C_3, \ldots, C_{g-1})$-free. To add a pendant vertex to a vertex $u \in V$, means to obtain a new graph from G by adding one more vertex and making it adjacent only to u. We denote the complete graph on r vertices by K_r. A graph is *chordal bipartite* if it is bipartite and every induced cycle has exactly four vertices.

We complete this section by providing some context for our work on (C_s, P_t)-free graphs. We comment that it can be seen as a natural continuation of investigations into the complexity of k-Colouring and List k-Colouring for P_t-free graphs (see [9]). The sharpest results are the following. Hoàng et al. [15] proved that, for all $k \geq 1$, List k-Colouring is p-time solvable on P_5-free graphs. Huang [16] proved that 4-Colouring is NP-complete for P_7-free graphs and that 5-Colouring is NP-complete for P_6-free graphs. Recently, Chudnovsky, Maceli and Zhong [5, 6] announced a p-time algorithm for solving 3-Colouring on P_7-free graphs. Broersma et al. [3] proved that List 3-Colouring is p-time solvable for P_6-free graphs. Golovach, Paulusma and Song [11] proved that List 4-Colouring is NP-complete for P_6-free graphs. These results lead to the following table (in which the open cases are denoted by "?").

	k-Colouring				k-Precolouring Extension				List k-Colouring			
	$k=3$	$k=4$	$k=5$	$k\geq 6$	$k=3$	$k=4$	$k=5$	$k\geq 6$	$k=3$	$k=4$	$k=5$	$k\geq 6$
$t \leq 5$	P	P	P	P	P	P	P	P	P	P	P	P
$t = 6$	P	?	NP-c	NP-c	P	?	NP-c	NP-c	P	NP-c	NP-c	NP-c
$t = 7$	P	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c
$t \geq 8$	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c

Table 1. The complexity of k-Colouring, k-Precolouring Extension and List k-Colouring for P_t-free graphs.

2 New Results for List Colouring

In this section we give two results on List 4-Colouring.

We first prove that List 4-Colouring is NP-complete for the class of $(C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4})$-free graphs. (We observe that $\overline{P_1 + 2P_2}$ is also known as the 5-vertex wheel and $\overline{P_1 + P_4}$ is sometimes called the gem or the 5-vertex fan.) This result strengthens an analogous result on List 4-Colouring of P_6-free graphs [11], and is obtained by a closer analysis of the hardness reduction used in the proof of that result. The reduction is from the problem Not-All-Equal 3-Sat with positive literals only which was shown to be NP-complete by Schaefer [26] and is defined as follows. The input I consists of a set $X = \{x_1, x_2, \ldots, x_n\}$ of variables, and a set $\mathcal{C} = \{D_1, D_2, \ldots, D_m\}$ of 3-literal clauses over X in which all literals are positive. The question is whether there exists a truth assignment for X such that each D_i contains at least one true literal and at least one false literal. We describe a graph J_I and 4-list assignment L that are defined using the instance I:

- a-type and b-type vertices: for each clause D_j, J_I contains two *clause components* D_j and D'_j each isomorphic to P_5. Considered along the paths the vertices in D_j are $a_{j,1}, b_{j,1}, a_{j,2}, b_{j,2}, a_{j,3}$ with lists of admissible colours $\{2,4\}, \{3,4\}, \{2,3,4\}, \{3,4\}, \{2,3\}$, respectively, and the vertices in D'_j are $a'_{j,1}, b'_{j,1}, a'_{j,2}, b'_{j,2}, a'_{j,3}$ with lists of admissible colours $\{1,4\}$, $\{3,4\}$, $\{1,3,4\}$, $\{3,4\}$, $\{1,3\}$, respectively.
- x-type vertices: for each variable x_i, J_I contains a vertex x_i with list of admissible colours $\{1,2\}$.
- For every clause D_j, its variables $x_{i_1}, x_{i_2}, x_{i_3}$ are ordered in an arbitrary (but fixed) way, and in J_I there are edges $a_{j,h}x_{i_h}$ and $a'_{j,h}x_{i_h}$ for $h = 1, 2, 3$.
- There is an edge in J_I from every x-type vertex to every b-type vertex.

See Figure 1 for an example of the graph J_I. In this figure, D_j is a clause with ordered variables $x_{i_1}, x_{i_2}, x_{i_3}$. The thick edges indicate the connection between these vertices and the a-type vertices of the two copies of the clause gadget. Indices from the labels of the clause gadget vertices have been omitted to aid clarity.

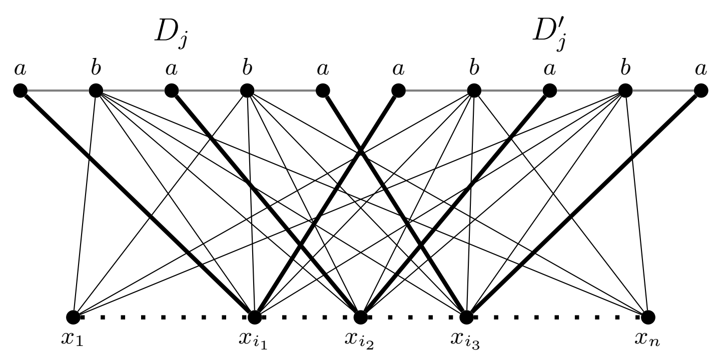

Fig. 1. An example of a graph J_I, as shown in [11]. Only the clause $D_j = \{x_{i_1}, x_{i_2}, x_{i_3}\}$ is displayed.

The following two lemmas are known.

Lemma 1 ([11]). *The graph J_I has a colouring that respects L if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal.*

Lemma 2 ([11]). *The graph J_I is P_6-free.*

We are now ready to prove the main result of this section.

Theorem 2. *The* List 4-Colouring *problem is* NP*-complete for the class of* $(C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4}, P_6)$*-free graphs.*

Proof. Lemma 1 shows that the List 4-Colouring problem is NP-hard for the class of graphs J_I, where $I = (X, \mathcal{C})$ is an instance of Not-All-Equal 3-Sat with positive literals only, in which every clause contains either two or three literals and in which each literal occurs in at most three different clauses. Lemma 2 shows that each J_I is P_6-free. As the List 4-Colouring problem is readily seen to be in NP, it remains to prove that each J_I contains no induced subgraph in the set $S = \{C_5, C_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4}\}$. For contradiction, we assume that some J_I has an induced subgraph H isomorphic to a graph in S.

First suppose that $H \in \{C_5, C_6\}$. The total number of x-type and b-type vertices can be at most 3, as otherwise H contains an induced C_4 or a vertex of degree at least 3. Because $|V(H)| \geq 5$ and the subgraph of H induced by its b-type and x-type vertices is connected, H must contain at least two adjacent a-type vertices. This is not possible.

Now suppose that $H = K_4$. Because the b-type and x-type vertices induce a bipartite graph, H must contain an a-type vertex. Every a-type vertex has degree at most 3. If it has degree 3, then it has two non-adjacent neighbours (which are of b-type); a contradiction.

Finally suppose that $H \in \{\overline{P_1 + 2P_2}, \overline{P_1 + P_4}\}$. Let u be the vertex that has degree 4 in H. Then u cannot be of a-type, because no a-type vertex has more than three neighbours in J_I. If u is of b-type, then every other vertex of H is either of a-type or of x-type, and because vertices of the same type are not adjacent, H must contain two a-type vertices and two x-type vertices. Then an a-type vertex is adjacent to two x-type vertices; a contradiction. Thus u must be of x-type, and so every other vertex of H is either of a-type or of b-type. Because vertices of the same type are non-adjacent, H must contain two a-type vertices and two b-type vertices. But then u is adjacent to two a-type vertices in the same clause-component. This is not possible. □

Our second result modifies the same reduction. From J_I and L, we obtain a new graph J'_I and list assignment L' by subdividing every edge between an a-type vertex and an x-type vertex and giving each new vertex the list $\{1, 2\}$. We say that these new vertices are of c-type.

Lemma 3. *The graph J'_I is P_8-free and chordal bipartite.*

Proof. We first prove that J'_I is P_8-free (but not P_7-free). Let P be an induced path in J'_I. If P contains no x-type vertex, then P contains vertices of at most one clause-component together with at most two c-type vertices. This means that $|V(P)| \leq 7$. If P contains no b-type vertex, then P can contain at most one x-type vertex (as any two x-type vertices can only be connected by a path that uses at least one b-type vertex). Consequently, P can have at most two a-type vertices and at most two c-type vertices. Hence, $|V(P)| \leq 5$ in this case. From now on assume that P contains at least one b-type vertex and at least one x-type vertex. Also note that P can contain in total at most three vertices of b-type and x-type.

First suppose that P contains exactly three vertices of b-type and x-type. Then these vertices form a 3-vertex subpath in P of types b, x, b or x, b, x. In both cases we can extend both ends of the subpath only by an a-type vertex and an adjacent c-type vertex, which means that $|V(P)| \leq 7$. Now suppose that P contains exactly two vertices of b-type and x-type. Because these vertices are of different type, they are adjacent and we can extend both ends of the corresponding 2-vertex subpath of P only by an a-type vertex and an adjacent c-type vertex. This means that $|V(P)| \leq 6$. We conclude that J'_I is P_8-free.

We now prove that J'_I is chordal bipartite. The graph J'_I is readily seen to be bipartite. Because J'_I is P_8-free, it contains no induced cycle on ten or more vertices. Hence, in order to prove that J'_I is chordal bipartite, it remains to show that J'_I is (C_6, C_8)-free.

For contradiction, let H be an induced subgraph of J'_I that is isomorphic to C_6 or C_8. Then H must contain at least one vertex of x-type and at least one vertex of b-type. The subgraph of H induced by its vertices of x-type and b-type is connected and has size at most 3. This means that the subgraph of H induced by its vertices of a-type and c-type is also connected and has size at least 3. This is not possible. We conclude that J'_I is chordal bipartite. $\square$

The following lemma can be proven by exactly the same arguments that were used to prove Lemma 1.

Lemma 4. *The graph J'_I has a colouring that respects L' if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal.*

Lemmas 3 and 4 imply the last result of this section.

Theorem 3. List 4-Colouring *is* NP*-complete for P_8-free chordal bipartite graphs.*

3 New Results for Precolouring Extension

In this section we give three results on k-Precolouring Extension.

Let $k \geq 4$. Consider the bipartite graph J'_I with its list assignment L' from Section 2 that was defined immediately before Lemma 3. The list of admissible colours $L'(u)$ of each vertex u is a subset of $\{1,2,3,4\}$. We add $k-|L'(u)|$ pendant vertices to u and precolour these vertices with different colours from $\{1,\ldots,k\}\setminus L'(u)$. This results in a graph J_I^k with a k-precolouring c_{W^k}, where W^k is the set of all the new pendant vertices in J_I^k.

Lemma 5. *For all $k \geq 4$, the graph J_I^k is P_{10}-free and chordal bipartite.*

Proof. Because J_I' is P_8-free and chordal bipartite by Lemma 3, and we only added pendant vertices, J_I^k is P_{10}-free and chordal bipartite. □

The following lemma can be proven by the same arguments that were used to prove Lemmas 1 and 4.

Lemma 6. *For all $k \geq 4$, the graph J_I^k has a k-colouring that is an extension of c_{W^k} if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal.*

Lemmas 5 and 6 imply the first result of this section which extends a result of Kratochvíl [21] who showed that 5-Precolouring Extension is NP-complete for P_{13}-free bipartite graphs.

Theorem 4. *For all $k \geq 4$, k-*Precolouring Extension *is* NP*-complete for the class of P_{10}-free chordal bipartite graphs.*

Here is our second result.

Theorem 5. *The* 4-Precolouring Extension *problem is* NP*-complete for the class of $(C_5, C_6, C_7, C_8, P_8)$-free graphs.*

Proof. Let J_I be the instance with list assignment L as constructed at the start of Section 2. Instead of considering lists, we introduce new vertices, which we precolour (we do not precolour any of the original vertices). For each clause component D_j we add five new vertices, s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$. We add edges $a_{j,1}s_j$, $a_{j,3}t_j$ and $a_{j,h}u_{j,h}$ for $h = 1,\ldots 3$. We precolour s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$ with colours 3, 4, 1, 1, 1, respectively. For each clause component D_j' we add five new vertices, s_j', t_j', $u_{j,1}'$, $u_{j,2}'$, $u_{j,3}'$. We add edges $a_{j,1}'s_j'$, $a_{j,3}'t_j'$ and $a_{j,h}'u_{j,h}'$ for $h = 1,\ldots 3$. We precolour s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$ with colours 3, 4, 2, 2, 2, respectively. Finally, we add two new vertices c_1, c_2, which we make adjacent to all x-type vertices, and two new vertices y_1, y_2, which we make adjacent to all b-type vertices. We colour c_1, c_2, y_1, y_2 with colours 3, 4, 1, 2, respectively. This results in a new graph J_I^*. Because y_1, y_2 can be viewed as x-type vertices and c_1, c_2 as b-type vertices, because every other new vertex is a pendant vertex and because J_I is (C_5, C_6, P_6)-free (by Theorem 2), we find that J_I^* is $(C_5, C_6, C_7, C_8, P_8)$-free. Moreover, our precolouring forces the list $L(v)$ upon every vertex v of J_I. Hence, J_I^* has a 4-colouring extending this precolouring if and only if J_I has a colouring that respects L. By Lemma 1 the latter is true if and only if I has a satisfying truth assignment in which each clause contains at least one true and at least one false literal. □

Broersma et al. [3] showed that 5-Precolouring Extension for P_6-free graphs is NP-complete. It can be shown that the gadget constructed in their NP-hardness reduction is (C_5, C_6)-free. By adding $k-5$ dominating vertices, precoloured with colours $6, \ldots, k$, to each vertex in their gadget, we can extend their result from $k = 5$ to $k \geq 5$. This leads to the following theorem.

Theorem 6. *For all $k \geq 5$, k-Precolouring Extension is NP-complete for (C_5, C_6, P_6)-free graphs.*

4 New Results for Colouring

The first result that we show in this section is that 4-Colouring is NP-complete for (C_3, P_{22})-free graphs. We will prove this by modifying the graph J_I^4 from Section 3. It improves the result of Golovach et al. [10] who showed that 4-Colouring is NP-complete for (C_3, P_{164})-free graphs.

First we review a well-known piece of graph theory. The *Mycielski construction* of a graph G is created by adding, for each vertex v of G, a new vertex that is adjacent to each vertex $N_G(v)$, and then adding a further vertex that is adjacent to each of the other new vertices. For example, the Mycielski construction of K_2 is a 5-cycle, and the Mycielski construction of a 5-cycle is the well-known Grötzsch graph. These examples are the first in an infinite sequence of graphs $M_2, M_3, \ldots$ where $M_2 = K_2$ and M_i, $i \geq 3$, is the Mycielski construction of M_{i-1}. The graph M_5 is displayed in Figure 2. As we will make considerable use of this graph, let us explain its construction carefully. Let us suppose that we start with M_3 where $V(M_3) = \{1, 2, 3, 4, 5\}$ and $E(M_3) = \{\{1, 2\}, \{2, 3\}, \{3, 5\}, \{5, 4\}, \{4, 1\}\}$ (note that, for clarity, we denote an edge between two vertices u and v by $\{u, v\}$ instead of uv). Then M_4 is made by adding each vertex i, $6 \leq i \leq 10$, and making it adjacent to the neighbours of vertex $i-5$ and to a further vertex 11. Finally M_5 is obtained by adding a vertex i, $12 \leq i \leq 22$, with the same neighbours as vertex $i-11$ and a further last new vertex, 23. Mycielski [23] showed that each M_k is C_3-free and has chromatic number k. Moreover, any proper subgraph of M_k is $(k-1)$-colourable (see for example [1]).

Let M' be the graph obtained from M_5 by removing the edge from 17 to 23. We need the following lemma.

Lemma 7. *Let ϕ be a 4-colouring of M'. Then $\phi(17) = \phi(23)$, and moreover, $\{\phi(2), \phi(4), \phi(11), \phi(17)\} = \{\phi(2), \phi(4), \phi(11), \phi(23)\} = \{1, 2, 3, 4\}$.*

Proof. If $\phi(17) \neq \phi(23)$, then ϕ is a 4-colouring of M_5, which is not possible. As $N_{M'}(17) = \{2, 4, 11\}$, we must have $\{\phi(2), \phi(4), \phi(11), \phi(17)\} = \{1, 2, 3, 4\}$ else there is a 4-colouring ϕ' which disagrees with ϕ only on 17 and so, again, is a 4-colouring of M_5. Thus we also have $\{\phi(2), \phi(4), \phi(11), \phi(23)\} = \{1, 2, 3, 4\}$. □

Let $T = \{t_1, t_2, t_3, t_4\} = \{2, 4, 11, 23\}$. We present three lemmas about induced paths in M' with endvertices in T. Proving each of these three lemmas is straightforward but tedious. We therefore omitted their proofs.

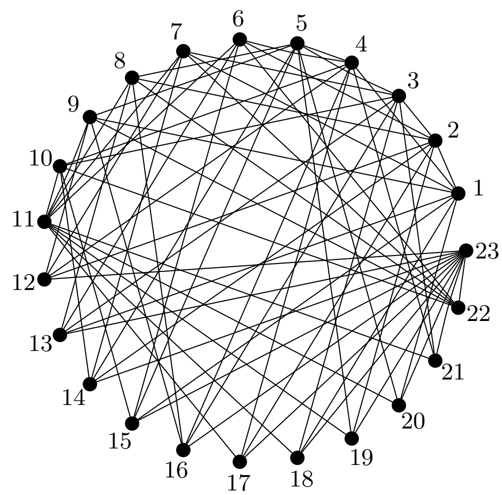

Fig. 2. The graph M_5: vertex sets $\{1,2\}$, $\{1,2,3,4,5\}$ and $\{1,2,3,4,5,6,7,8,9,10,11\}$ induce M_2, M_3 and M_4, respectively.

Lemma 8. *Every induced path in M' with both endvertices in T contains at most 7 vertices.*

Lemma 9. *Every induced path in M' with an endvertex in T contains at most 8 vertices.*

Lemma 10. *M' contains no induced subgraph isomorphic to $P_8 + P_1$ such that each of the two paths has an endvertex in T. Also, M' contains no induced subgraph isomorphic to $2P_7$ such that each of the two paths has an endvertex in T.*

For a graph $G = (V, E)$ and subset $U \subseteq V$, we let $G - U$ denote the graph obtained from G after removing all vertices in U.

To prove our result we also use the graphs J'_I and J^4_I defined in Sections 2 and 3, respectively. Let A, B, C, X denote the sets of of a-type, b-type, c-type and x-type vertices in J'_I, respectively. Note these sets also exist in J^4_I. We need an upper bound on the length of an induced path in some specific induced subgraphs of J'_I and J^4_I.

Lemma 11. *Every induced path in $J'_I - C$ that starts with an a-type or x-type vertex has at most 5 vertices.*

Proof. Let $P = v_1v_2, \dots v_r$ be a maximal induced path in J'_I. Suppose that v_1 is of a-type. Then, as P contains no c-type vertices, we find that $v_2 = b$. Consequently, $v_3 = a$ or $v_3 = x$. In both cases we must have $v_4 = b$ and $v_5 = a$. Afterward we cannot extend P any further. Suppose that v_1 is of x-type. Because there is no c-type vertex in P, we find that $v_2 = b$. Hence, $v_3 = a$ or $v_3 = x$. In both cases, P cannot be extended further. We conclude that $r \leq 5$. $\square$

Lemma 12. *Every induced path in $J_I^4 - B$ has*

(i) at most 7 vertices;
(ii) at most 6 vertices if it contains only one pendant vertex of $J_I^4 - B$;
(iii) at most 5 vertices if it contains no pendant vertex of $J_I^4 - B$.

Proof. We observe that $J_I^4 - B$ is a forest in which each tree can be constructed as follows. Take a star. Subdivide each of its edges exactly once. Afterward add one or more pendant vertices to each vertex. The tree obtained is P_8-free and every induced P_7 contains two pendant vertices as its end-vertices. Moreover, every induced P_6 has at least one pendant vertex. □

We will now modify the (bipartite) graph J_I^4 with the precolouring c_{W^4} in the following way.

- Remove all vertices that are pendant to any vertex in $B \cup C$.
- Add a copy of M'. Write $t_1 = 2$, $t_2 = 4$, $t_3 = 11$ and $t_4 = 23$.
- Let S be the set of vertices pendant to any vertex in $A \cup X$. For each $v \in S$ do as follows. If $c_{W^4}(v) = i$ then add the edge vt_j for all $j \in \{1, 2, 3, 4\} \setminus \{i\}$.
- Add an edge between every vertex in B and t_i for $i = 1, 2$.
- Add an edge between every vertex in C and t_i for $i = 3, 4$.

We call the resulting graph J_I^*. Note that J_I^* is not P_{21}-free in general. In order to see this we say that the pendant vertices of $B \cup C$ are of type p and that the vertices of T are of type t and then take an induced 21-vertex path with vertices of type

$$a - c - x - c - a - p - t - p - a - c - x - c - a - p - t - p - a - c - x - c - a.$$

Note that such a path only uses two vertices of M', namely t_1 and t_2 (as t_3 and t_4 are adjacent to all vertices of c-type). As such, trying to optimize Lemmas 8–10 (which we believe is possible) does not help us with improving our result. In the following lemma we show that this length is maximum.

Lemma 13. *The graph J_I^* is (C_3, P_{22})-free.*

Proof. Because $S \cup B \cup C$ and $\{t_1, t_2, t_3, t_4\}$ are both independent sets and the graphs M' and J_I^4 are bipartite, J_I^* is C_3-free. Below we show that J_I^* is P_{22}-free. Let P be an induced path in J_I^*. Let α be the number of vertices of $\{t_1, t_2, t_3, t_4\}$ in P.

Case 1. $\alpha = 0$.
Then either $P \subseteq J_I^4$, and so $|V(P)| \leq 9$ by Lemma 5, or $P \subseteq M'$, and so $|V(P)| \leq 21$.

Case 2. $\alpha = 1$.
Then $P = P_L t_i P_R$ for some $1 \leq i \leq 4$, where the subpaths P_L and P_R are each fully contained in either M' or J_I^4. If both of them are contained in M', then $P \subseteq M'$ and so $|V(P)| \leq 21$. If one of them is contained in M' and the other

one is contained in J_I^4, then $|V(P)| \leq 8+9 = 17$ by Lemmas 5 and 9. Otherwise both of them are contained in J_I^4 and so $|V(P)| \leq 9+1+9 = 19$ by Lemma 5.

Case 3. $\alpha = 2$.
Then $P = P_L t_i P_M t_j P_R$ for some $i, j \in \{1,2,3,4\}$, where each of P_L, P_M and P_R is fully contained in either M' or J_I^4. We need the following claim.

Claim 1. $|V(P_L)| \leq 6$ *and* $|V(P_R)| \leq 6$.

We prove Claim 1 as follows. By symmetry it suffices to show it for P_L only. First assume that P_L is contained in J_I^4. We distinguish two cases.

Case a. $t_i = t_1$.
Let t_1^- be the right endvertex of P_L. First assume that $t_1^- \notin B$. Then t_1^- must be in S by the construction of J_I^*. Because t_1 is adjacent to every vertex in B, we have $V(P_L) \cap B = \emptyset$. Hence, $P_L \subseteq J_I^4 - B$. We also have $|V(P_L) \cap S| = 1$, as P is induced and each vertex in S is adjacent to three vertices of T. Then, by Lemma 12, we obtain $|V(P_L)| \leq 6$. Now assume $t_1^- \in B$. Then $|V(P_L) \cap S| = 0$ for the same reason as before. Hence $P_L - \{t_1^-\}$ contains only non-pendant vertices of $J_I^4 - B$. Then, by Lemma 12, we obtain $|V(P_L)| \leq 5+1 = 6$.

Case b. $t_i = t_3$.
Let t_3^- be the right endvertex of P_L. If $t_j \in \{t_1, t_2\}$ then $P_L \cap B = \emptyset$, and so P_L is an induced path of $J_I^4 - B$. By Lemma 12 we find that $|V(P_L)| \leq 6$. Now assume that $t_j = t_4$. Since t_4 is adjacent to all vertices of C, we have $V(P_L) \cap C = \emptyset$. Hence, by construction, we find that $t_3^- \in S$. In fact we have $V(P) \cap S = \{t_3^-\}$, as P is induced and each vertex in S is adjacent to three vertices of T. Thus, $P_L - \{t_3^-\}$ is contained in $J_I' - C$. As $t_3^- \in S$, its neighbour on P_L must be in $A \cup X$ (if this neighbour exists). Then, by Lemma 11, we find that $|V(P_L)| \leq 5+1 = 6$.

Now suppose that P_L is contained in M'. Using Lemma 10 with respect to the paths $P_L t_i$ and t_j we conclude that $|V(P_L)| \leq 6$. This completes the proof of Claim 1.

By Claim 1 we have that $|V(P_L)| \leq 6$ and $|V(P_R)| \leq 6$. If $|V(P_M)| \leq 7$ this means that $|V(P)| \leq 6+1+7+1+6 = 21$ and we are done. So, it suffices to prove that $|V(P_M)| \leq 7$, as we will do below.

If P_M is contained in M', then by Lemma 8 we have that $|V(P_M)| \leq 5$. Assume that $P_M \subseteq J_I^4$. First suppose that $\{t_i, t_j\} = \{t_1, t_2\}$. If P_M contains a b-type vertex, then P_M must be a single b-type vertex and thus $|V(P_M)| = 1$. Otherwise we have $P_M \subseteq J_I^4 - B$, and hence $|V(P_M)| \leq 7$ by Lemma 12.

Now suppose that $\{t_i, t_j\} = \{t_3, t_4\}$. If P_M contains a c-type vertex, then P_M must be a single c-type vertex and thus $|V(P_M)| = 1$. Otherwise both endvertices of P_M are in S and all internal vertices of P_M are contained in $J_I' - C$. Because every neighbour of a vertex in S in J_I' belongs to $A \cup X$, we find that $|V(P_M)| \leq 1+5+1 = 7$ by Lemma 11.

We observe that the pairs $\{t_1, t_2\}$ and $\{t_3, t_4\}$ are symmetric in terms of the edges between them to B and C. Moreover, t_1 and t_2 can be seen as symmetric, and the same holds for t_3 and t_4. This has the following consequence. Suppose that $\{t_i, t_j\} \notin \{\{t_1, t_2\}, \{t_3, t_4\}\}$. Then we may assume without loss of generality that $\{t_i, t_j\} = \{t_1, t_3\}$, say $P = P_L t_1 P_M t_3 P_R$.

If the left endvertex of P_M is not in B, then $P_M \subseteq J_I^4 - B$ and so $|V(P_M)| \leq 7$ by Lemma 12. Similarly, if the right endvertex of P_M is not in C then all vertices except the left endvertex of P_M are in $J_I' - C$. In particular, the right endvertex of P_M belongs to S in that case, and as such has its neighbour on P_M in $A \cup X$ (if this neighbour exists). Then $|V(P_M)| \leq 1 + 5 = 6$ by Lemma 11. Therefore we may assume that P_M starts with a vertex in B, ends at a vertex in C. As P is induced and each vertex in S is adjacent to three vertices of T, we find that P_M contains no vertices of S. Moreover, no internal vertex of P_M belongs to $B \cup C$, as these vertices would be adjacent to one of $\{t_1, t_3\}$. Hence all internal vertices of P_M are in $A \cup X$. Then P_M only has one internal vertex, which is either in A or in X. Hence, $|V(P_M)| = 3 \leq 7$. This completes the proof of Case 3.

Case 4. $\alpha = 3$.
Then $P = P_L t_h P_M^1 t_i P_M^2 t_j P_R$ for some $h, i, j \in \{1, 2, 3, 4\}$, where the subpaths P_L, P_M^1, P_M^2 and P_R are each fully contained in either M' or J_I^4. As P is induced and each vertex in S is adjacent to three vertices of T. we find that $|V(P_M^i)| = 1$ if P_M^i $(i = 1, 2)$ contains a vertex of S. We observe that Claim 1 is also valid here (as exactly the same arguments can be used). Hence, $|V(P_L)| \leq 6$ and $|V(P_R)| \leq 6$. By the aforementioned symmetry relations between vertices and vertex pairs of $\{t_1, t_2, t_3, t_4\}$, we may assume without loss of generality that $t_1, t_3, t_4 \in P$. According to the relative positions among these three vertices on the path P we have the following two subcases.

Case 4.1 $P = P_L t_1 P_M^1 t_3 P_M^2 t_4 P_R$.
We first prove that either P_M^1 is contained in M' or that $|V(P_M^1)| = 1$. Suppose that P_M^1 is not contained in M'. The right endvertex of P_M^1 cannot be in B due to the presence of t_4. Hence, it must be in S. As observed above, in that case we have $|V(P_M^i)| = 1$. We now prove that either P_M^2 is contained in M' or that $|V(P_M^2)| = 1$. Suppose that P_M^2 is not contained in M'. Then P_M^2 is contained in J_I^4. As mentioned above, $|V(P_M^2)| = 1$ if P_M^2 contains a vertex of S. So, assume P_M^2 contains no vertex of S. Then P_M^2 must contain a vertex of C and consequently cannot contain any other vertex. Hence $|V(P_M^2)| = 1$.

Now, if each P_M^i is contained in M' then $|V(P)| \leq 6+7+6 = 19$ by Lemma 8 and the aforementioned fact that $|V(P_L)| \leq 6$ and $|V(P_R)| \leq 6$. Suppose that P_M^1 is not contained in M'. Then $|V(P_M^1)| = 1$. If P_M^2 is contained in M' then $t3 P_M^2 t_4$ has at most seven vertices by Lemma 10. Thus, $|V(P)| \leq 6 + 1 + 1 + 7 + 6 = 21$. Otherwise, $|V(P_M^2)| = 1$, and consequently, $|V(P)| \leq 17$. If P_M^2 is not contained in M', we can repeat the arguments.

Case 4.2 $P = P_L t_3 P_M^1 t_1 P_M^2 t_4 P_R$.
Note that P contains no vertices of B and recall that $|V(P_M^i)| = 1$ if P_M^i $(i = 1, 2)$ contains a vertex of S. Hence, P contains no vertices of C either, and the claim that either P_M^i is contained in M' or $|V(P_M^i)| = 1$ is still true for $i = 1, 2$. By repeating the arguments as in Case 4.1 we find that $|V(P)| \leq 21$.

Case 5. $\alpha = 4$.
Then $P = P_L t_h P_M^1 t_i P_M^2 t_j P_M^3 t_k P_R$ for $\{h, i, j, k\} = \{1, 2, 3, 4\}$, where the subpaths P_L, P_M^1, P_M^2, P_M^3 and P_R are each fully contained in either M' or J_I^4. Note that $V(P) \cap S = \emptyset$ as P is induced and each vertex in S is adjacent to three vertices of T.

By symmetry, it suffices to consider the following three cases.

Case 5.1. $P = P_L t_1 P_M^1 t_3 P_M^2 t_2 P_M^3 t_4 P_R$.
Then P cannot contain any vertex of $B \cup C$. Consequently, as P contains no vertex from S either, P is a path in M'. Hence, $|V(P)| \leq 21$.

Case 5.2. $P = P_L t_1 P_M^1 t_3 P_M^2 t_4 P_M^3 t_2 P_R$.
Then P contains no vertex of C, and only P_M^2 may contain a vertex of $B \cup C$, which must be in B. Hence, as P contains no vertex of S, all other three subpaths are fully contained in M'. If P_M^2 contains no vertex of B, then P is a path in M' and hence $|V(P)| \leq 21$. Otherwise, $|V(P_M^2)| = 1$. Then $|V(P)| \leq 8 + 1 + 8 = 17$ by Lemma 9.

Case 5.3. $P = P_L t_1 P_M^1 t_2 P_M^2 t_3 P_M^3 t_4 P_R$.
As P has no vertex of S, all paths P_L, P_M^2, P_R must belong to M', whereas P_M^1 and P_M^3 are either a single vertex (not in M') or contained in M'. If both of P_M^1 and P_M^3 are contained in M', then P is contained in M', and thus $|V(P)| \leq 21$. If exactly one of P_M^1 and P_M^3 is contained in M', then $|V(P)| \leq 8+1+8 = 17$ by Lemma 9. Thus $|V(P_M^1)| = 1$ and $|V(P_M^3)| = 1$. By Lemma 10 we find that P_L, P_M^2 and P_L each have at most seven vertices. Moreover, Lemma 10 also implies that the sum of the order of any two of these subpaths is at most 13. Hence, $|V(P)| \leq 3 \times 13/2 + 2 = 21.5$. Thus, $|V(P)| \leq 21$. This completes the proof of Case 5, and as such we have proven the lemma. □

The following lemma follows from the way we constructed J_I^* from J_I^4.

Lemma 14. *The graph J_I^* has a 4-colouring if and only if J_I^4 has a 4-colouring that is an extension of c_{W^k}.*

Combining Lemmas 6, 13 and 14 gives us the main result of this section.

Theorem 7. 4-Colouring *is* NP*-complete for (C_3, P_{22})-free graphs.*

To prove the final result of this section we need the following classical result of Erdös [7] as a lemma (we follow the same approach as that used by Král' et al. [20] and Kamiński and Lozin [18] for proving that for all $k \geq 3$ and $s \geq 3$, k-Colouring is NP-complete for $(C_3, \ldots, C_s)$-free graphs).

Lemma 15 ([7]). *For every pair of integers g and k, there exists a graph with girth g and chromatic number k.*

This final result extends a result of Golovach et al. [10] who proved that for all $s \geq 6$, there exists a constant t^s such that 4-Colouring is NP-complete for $(C_5, \ldots, C_{s-1}, P_{t^s})$-free graphs. It is also known that, for all $k \geq 3$ and all $s \geq 4$, k-Colouring is NP-complete for graphs of girth at least s, or equivalently,

$(C_3, \ldots, C_{s-1})$-free graphs. This has been shown by Král', Kratochvíl, Tuza, and Woeginger [20] for the case $k = 3$ and by Kamiński and Lozin [18] for the case $k \geq 4$. On the other hand, Golovach et al. [10] showed that for all $k \geq 1$ and $r \geq 1$, even List k-Colouring is p-time solvable for (C_4, P_r)-free graphs (also see Theorem 1), and thus for $(C_3, C_4, \ldots, C_{s-1}, P_r)$-free graphs for all $s \geq 5$. As such, our new hardness result is best possible and can be seen as an analog (for $k \geq 4$) of the aforementioned result shown by Král' et al. [20] and Kamiński and Lozin [18].

Theorem 8. *For all $k \geq 4$ and $s \geq 6$, there exists a constant t_k^s such that k-Colouring is NP-complete for $(C_3, C_5, \ldots, C_{s-1}, P_{t_k^s})$-free graphs.*

Proof. Let $k \geq 4$ and $s \geq 6$. By Lemma 15, there exists an edge-minimal graph F with chromatic number $k+1$ and girth s. Let pq be an edge in F and let $F-pq$ denote the graph obtained from F after removing pq. Then, by definition, $F-pq$ has at least one k-colouring, and moreover, p and q receive the same colour in every k-colouring of $F-pq$. We introduce a new vertex q^* that we make adjacent only to q. Call the resulting graph F'. Then F' has at least one k-colouring, and moreover, p and q^* receive a different colour in every k-colouring of F'. An F'-*identification* of some edge uv in a graph G is the following operation: delete the edge uv and add a copy of F' between u and v by identifying vertices u and v with p and q^*, respectively (we call these two new vertices u and v again).

Now consider the graph J_I^4. We take a complete graph on k new vertices $r_1, \ldots, r_k$. Recall that we had defined a precolouring c_{W^4} for the subset $W^4 \subseteq V(J_I^4)$. We add an edge between a vertex r_i and a vertex $u \in W^4$ if and only if $c_{W^4}(u) \neq i$. Afterward we perform an F'-identification of every edge between two vertices r_i and r_j and on every edge between a vertex r_i and a vertex in W^4. Let G_I^4 be the resulting graph.

We observe that G_I^4 is not C_4-free. However, because J_I^4 is chordal bipartite by Lemma 5 and because we performed appropriate F'-identifications, G_I^4 is $(C_3, C_5, \ldots, C_{s-1})$-free. Let Q be an induced path in G_I^4. Let $h = |V(Q)| \cap \{r_1, \ldots, r_k\}$. Then we can write $Q = Q_1 r_{i_1} Q_2 r_{i_2} \cdots Q_h r_{i_h} Q_{h+1}$, where the vertices of each Q_i all belong either to an F'-copy or to J_k^4. Because $|F'|$ is a constant that depends only on k and s, and J_I^4 is P_{10}-free by Lemma 5, we find that there exists a constant t_k^s, only depending on k and s, such that Q has length at most t_k^s. Hence, G_I^4 is $P_{t_k^s}$-free.

By construction, G_I^4 has a k-colouring if and only if J_I^4 has a 4-colouring that is an extension of c_{W^4}. We are left to apply Lemma 6 and to recall that Not-All-Equal 3-SAT with positive literals is NP-complete. □

Remark 1. Theorem 8 implies Theorem 1 (iii).6; we can choose $t_k = t_k^6$ for example. We claim that a slight modification of the construction used in the proof of Theorem 8 gives us a better upper bound for t_k. Instead of using an edge-minimal graph F with chromatic number $k+1$ and girth s, we take the (C_3-free) Mycielski graph M_{k+1}. Following the proof of Theorem 8 we pick an edge pq of M_{k+1} and obtain a modified graph M'_{k+1}. The graph obtained from J_I^4 after performing the M'_{k+1}-identifications is C_3-free. Let $Q_1 r_{i_1} Q_2 r_{i_2} \cdots Q_h r_{i_h} Q_{h+1}$ be an induced path in this graph where the vertices of each Q_i all belong either to an M'_{k+1}-copy or to J_k^4. Because $|V(M_k)| = 3 \cdot 2^{k-2} - 1$ for all $k \geq 2$ and J_I^4 is P_{10}-free, we find that, for all $k \geq 5$, $t_k \leq k + (k+1)(3 \cdot 2^{k-1} - 1)$.

5 Proof of Theorem 1

To prove Theorem 1 we need first to discuss some additional results. Kobler and Rotics [19] showed that for any constants p and k, List k-Colouring is p-time solvable on any class of graphs that have clique-width at most p, assuming that a p-expression is given. Oum [24] showed that a $(8^p - 1)$-expression for any n-vertex graph with clique-width at most p can be found in $O(n^3)$ time. Combining these two results leads to the following theorem.

Theorem 9. *Let $\mathcal{G}$ be a graph class of bounded clique-width. For all $k \geq 1$,* List k-Colouring *can be solved in p-time on $\mathcal{G}$.*

We also need the following result due to Gravier, Hoáng and Maffray [12] who slightly improved upon a bound of Gyárfás [13] who showed that every (K_s, P_t)-free graph can be coloured with at most $(t-1)^{s-2}$ colours.

Theorem 10 ([12]). *Let $s, t \geq 1$ be two integers. Then every (K_s, P_t)-free graph can be coloured with at most $(t-2)^{s-2}$ colours.*

We are now ready to prove Theorem 1 by considering each case. For each we either refer back to an earlier result, or give a reference; the results quoted can clearly be seen to imply the statements of the theorem.

Proof of Theorem 1. We first consider the intractable cases of List k-Colouring and note that (i).1 follows from Theorem 3, and Theorem 2 implies that List 4-Colouring is NP-complete for the class of (C_5, C_6, P_6)-free graphs which proves (i).2. We now consider the tractable cases. Erdös, Rubin and Taylor [8] and Vizing [28] observed that 2-List Colouring is p-time solvable on general graphs implying (i).3. Broersma et al. [3] showed that List 3-Colouring is p-time solvable for P_6-free graphs from which we can infer (i).4 and (i).6. Golovach et al. [10] proved that for all $k, r, s, t \geq 1$, List k-Colouring can be solved in linear time for $(K_{r,s}, P_t)$-free graphs. By taking $r = s = 2$, we obtain (i).5 and (i).8. The class of (C_3, P_6)-free graphs was shown to have bounded clique-width by Brandstädt, Klembt and Mahfud [2]; using Theorem 9 we see that List k-Colouring is p-time solvable on (C_3, P_6)-free graphs for all $k \geq 1$ demonstrating (i).7. Hoàng, Kamiński, Lozin, Sawada, and Shu [15] proved that for all $k \geq 1$, List k-Colouring is p-time solvable on P_5-free graphs proving (i).9.

We now consider k-Precolouring Extension. The tractable cases all follow from the results on List k-Colouring just discussed. So we are left to consider the NP-complete cases. Theorem 4 implies (ii).1 and (ii).6. Theorems 5 and 6 imply (ii).4 and (ii).7, respectively. And (ii).2, (ii).3 and (ii).5 follow immediately from corresponding results for k-Colouring proved by Hell and Huang [14].

Finally, we consider k-Colouring; first the NP-complete cases. Theorem 7 gives us (iii).1. Recall that Theorem 8 implies Theorem 1 (iii).6 (see Remark 1 in Section 4). Hell and Huang [14] proved all the other NP-completeness subcases. Chudnovsky, Maceli and Zhong [5, 6] announced that 3-Colouring is p-time solvable on P_7-free graphs, which gives us (iii).10 and (iii).12. Chudnovsky, Maceli, Stacho and Zhong [4] announced that 4-Colouring is p-time solvable for (C_5, P_6)-free graphs, which gives us (iii).15. Theorem 10 gives us (iii).17. All other tractable cases follow from the corresponding tractable cases for List k-Colouring. □

6 Open Problems

From Theorem 1, we see that the following cases are open in the classification of the complexity of graph colouring problems for (C_s, P_t)-free graphs (recall that t_k is a constant only depending on k).

(i) For List k-Colouring the following cases are open:
- $k = 3$, $s = 3$ and $t \geq 7$
- $k = 3$, $s \geq 5$ and $t \geq 7$
- $k \geq 4$, $s = 3$ and $t = 7$.

(ii) For k-Precolouring Extension the following cases are open:
- $k = 3$, $s = 3$ and $t \geq 7$
- $k = 3$, $s \geq 5$ and $t \geq 7$
- $k = 4$, $s = 3$ and $7 \leq t \leq 9$
- $k = 4$, $s \geq 5$ and $t = 6$
- $k = 4$, $s = 7$ and $t = 7$
- $k \geq 5$, $s = 3$ and $7 \leq t \leq 9$.

(iii) For k-Colouring the following cases are open:
- $k = 3$, $s = 3$ and $t \geq 8$
- $k = 3$, $s \geq 5$ and $t \geq 8$
- $k = 4$, $s = 3$ and $7 \leq t \leq 21$
- $k = 4$, $s \geq 6$ and $t = 6$
- $k = 4$, $s = 7$ and $7 \leq t \leq 8$
- $k \geq 5$, $s = 3$ and $k + 3 \leq t \leq t_k - 1$
- $k \geq 5$, $s = 5$ and $t = 6$.

Besides solving these missing cases (and the missing cases from Table 1) we pose the following three problems specifically. First, does there exist a graph H and an integer $k \geq 3$ such that List k-Colouring is NP-complete and k-Colouring is p-time solvable for H-free graphs? Theorem 1 shows that if we forbid two induced subgraphs then the complexity of these two problems *can* be different: take $k = 4$, $H_1 = C_5$ and $H_2 = P_6$. Second, is List 4-Colouring NP-complete for P_7-free bipartite graphs? This is the only missing case of List 4-Colouring for P_t-free bipartite graphs due to Theorems 1 and 3. Third, what is the computational complexity of List 3-Colouring and 3-Precolouring Extension for chordal bipartite graphs?

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